

Hankel Determinants H_2 and H_3 for the Logarithmic Coefficients of a Subclass of Close-to-Star Functions

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Abstract

This paper investigates the logarithmic coefficients of a subclass of close-to-star functions. We establish precise bounds of the logarithmic coefficients and the Hankel determinants $H_{2,1}, H_{2,2}$ and $H_{3,1}$ associated with the class $\mathcal{CST}_0(g)$, where $g(z)=z$. Additionally, we derive sharp estimates for the Zalcman functional $J_{3,3}(f)$ within the same class.

Keywords: Univalent functions, close-to-star functions, Hankel determinant, logarithmic coefficients, Zalcman functional, Carathéodory class.

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1. Introduction

This paper lies within the broader framework of geometric function theory, particularly the theory of analytic and univalent functions in the unit disk.

Specifically, it focuses on the subclass of close-to-star functions, which are generalizations of starlike functions and are fundamental in understanding the geometric properties of analytic functions.

Let $\mathcal{H}$ be the class of analytic functions in the unit disk D :
 $= \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in D), \quad (1.1)$$

and $\mathcal{A}$ be its subclass of all f normalized by $f(0) = 0, f'(0) = 1$. By $\mathcal{S}$ we denote the subclass of $\mathcal{A}$ of univalent functions, $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ of all starlike functions, namely, $f \in \mathcal{S}^*$ if $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in D. \quad (1.2)$$

A function $f \in \mathcal{A}$ is called close-to-star if there exists $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$ such

that

$$\operatorname{Re} \frac{e^{i\beta} f(z)}{g(z)} > 0, \quad z \in D. \quad (1.3)$$

Denote by $\mathcal{CST}$ the class of all close-to-star functions introduced by Reade (1955).

Note that $f \in \mathcal{CST}$ if and only if a function

$$F(z) = \int_0^z \frac{f(t)}{t} dt, z \in D,$$

is close-to-convex.

The class of close-to-star functions and its subclasses were intensively studied by various authors (e.g., MacGregor (1963), Sakaguchi (1964), Causey and Merkes (1970); for further references, see the reference by Duren (1983). Given $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$, let $\mathcal{CST}_\beta(g)$ be the subclass of $\mathcal{CST}$ of all f satisfying (1.1). In this paper, we deal with the class $\mathcal{CST}_0(g)$ where $g(z) = z, z \in D$, whose elements f given (1.1) satisfy the following condition:

$$\operatorname{Re} \frac{f(z)}{z} > 0, z \in D. \quad (1.4)$$

Also, we provide the sharp estimates of logarithmic coefficients of the Hankel determinant $H_{2,1}, H_{2,2}$ and $H_{3,1}$ and Zalcman functional $J_{2,4}(f)$ for the class $\mathcal{CST}_0(g)$.

Since the class $\mathcal{CST}_0(g)$ has a representation using the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in D, \quad (1.5)$$

having a positive real part in D , the coefficients of functions in $\mathcal{CST}_0(g)$ can be represented in terms of the coefficients of functions in $\mathcal{P}$. Therefore, to get the upper bounds of the considered functionals our computing is based on parametric formulas for the second, third and fourth coefficients in $\mathcal{P}$. However, the class is rotation non-invariant. Thus, to solve the discussed problems, we will apply a general formula for c_3 and c_4 recently obtained by Libera and Złotkiewicz (1983) and Kwon, et al. (2018).

The q -th Hankel determinant for analytic functions was first given by Pommerenke (1966) and is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}, \quad (1.6)$$

where $a_1 = 1, q, n \in \mathbb{N} = \{1, 2, \dots\}$.

In particular, we have

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, (a_1 = 1, n = 1, q = 2), \quad (1.7)$$

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2, (n = 2, q = 2), \quad (1.8)$$

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3 a_5 - a_3^3 - a_4^2 - a_2^2 a_5 + 2 a_2 a_3 a_4 = a_3 H_{2,2}(f) + a_4 I + a_5 H_{2,1}(f), \quad (1.9)$$

where $I = a_2 a_3 - a_4$,

and

$$H_{2,3}(f) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = a_3 a_5 - a_4^2, (n = 3, q = 2). \quad (1.10)$$

Recently, many authors examined the second and the third Hankel determinants

$H_{2,2}(f)$ and $H_{3,1}(f)$ over selected subclasses of $\mathcal{A}$, particularly of $\mathcal{S}$ (See Cho et al. (2017), (2018)) and Kowalczyk et al. (2018)).

Logarithmic coefficients are one of the topics recently being of research interest by various authors (Girela (2000), Thomas (2016), Ali et al. (2018), Kumar and Vasudevarao (2018) and Cho et al. (2020)).

For $f \in \mathcal{S}$ let

$$F_f(z) = \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n, z \in D, \log 1 = 0. \quad (1.11)$$

The numbers $\gamma_n := \gamma_n(f)$ are called logarithmic coefficients of f . It is well known

that the logarithmic coefficients play a crucial role in the Milin conjecture, (See Milin (1971) and Duren (1983)).

Taking the exponential of both sides of (1.11), and differentiating logarithmically, we obtain

$$\gamma_1 = \frac{1}{2} a_2, \quad (1.12)$$

$$\gamma_2 = \frac{1}{2} \left(a_3 - \frac{1}{2} a_2^2 \right), \quad (1.13)$$

$$\gamma_3 = \frac{1}{2} \left(a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right), \quad (1.14)$$

$$= \frac{1}{2} \left(a_5 - a_2 a_4 + a_2^2 a_3 - \frac{1}{2} a_3^2 - \frac{1}{4} a_2^4 \right), \quad (1.15)$$

$$\gamma_5 = \frac{1}{2} \left(a_6 - a_2 a_5 - a_3 a_4 + a_2^2 a_4 + a_2 a_3^2 - a_3 a_2^3 + \frac{1}{5} a_2^5 \right). \quad (1.16)$$

Lecko and Sim (2019) studied logarithmic coefficients on families related to close-to-star functions, while Hermtian Toeplitz determinants of the second and third order for classes of close-to-star functions were studied by Jastrzebski *et al.* (2020).

Very recently, Kowalczyk and Lecko (2022) introduced the research study of the Hankel determinant $H_{q,n}(F_f/2)$ which entries are logarithmic coefficients of f i.e., $H_{q,n}(F_f/2)$

$$= \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q-2} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}. \quad (1.17)$$

Kowalczyk and Lecko (2022) obtained sharp bounds for $H_{2,1}(F_f/2)$ for the classes of starlike and convex of order α . The problem of computing the sharp bounds of Hankel determinant on logarithmic coefficients of bounded turning associated with a petal-shaped domain considered by Shi *et al.* (2022).

Guo *et al.* (2024) investigated the logarithmic coefficients for a subclass of close-to-star functions, specifically the subclass $\mathcal{ST}(1)$. Additionally, he derives sharp estimates for the Zalcman functional $\mathcal{J}_{2,4}(f)$ within the same class.

Recently, Seker *et al.* (2025) investigated the logarithmic coefficients for the Class of Bounded Turning Functions Associated with Lune domain.

In this study, we investigate the estimation of Zalcman functional on the logarithmic coefficients, second and third Hankel determinants for the class $\mathcal{CST}_0(g)$ related to the identity function with the determinant entry of logarithmic coefficients. Also, we obtain the sharp estimate of Zalcman inequality for the class $\mathcal{CST}_0(g)$.

This study contributes to the deeper understanding of close-to-star functions by employing advanced tools such as logarithmic Hankel determinants. It not only enhances the theoretical framework of univalent function theory but also opens avenues for applications in applied mathematics, complex geometry, and mathematical physics.

2. Preliminary Results

In this section, we give some lemmas to prove our main results.

Lemma 2.1. (See Libera & Złotkiewicz, 1983) Let $p \in \mathcal{P}$ and be of the form (1.6). Then,

$$c_1 = 2\varepsilon_1, \quad (2.1)$$

$$\begin{aligned} c_2 &= 2\varepsilon_1^2 \\ &+ 2(1 - \varepsilon_1^2)\varepsilon_2, \end{aligned} \quad (2.2)$$

$$\begin{aligned} c_3 &= 2\varepsilon_1^3 + 4(1 - \varepsilon_1^2)\varepsilon_1\varepsilon_2 - 2(1 - \varepsilon_1^2)\varepsilon_1\varepsilon_2^2 \\ &+ 2(1 - \varepsilon_1^2)(1 - (|\varepsilon_2|^2))\varepsilon_3, \end{aligned} \quad (2.3)$$

$$\begin{aligned} c_4 &= 2\varepsilon_1^4 + 2(1 - \varepsilon_1^2)\varepsilon_2[\varepsilon_1^2(\varepsilon_2^2 - 3\varepsilon_2 + 3) + \varepsilon_2] \\ &- 2(1 - \varepsilon_1^2)(1 - (|\varepsilon_2|^2)) \\ &[2\varepsilon_1(\varepsilon_2 - 1)]\varepsilon_3 + \varepsilon_2\varepsilon_3^2 \\ &- (1 - (|\varepsilon_3|^2))\varepsilon_3, \end{aligned} \quad (2.4)$$

for some ε_2 and ε_3 such that $|\varepsilon_2| \leq 1$ and $|\varepsilon_3| \leq 1$ and $|\varepsilon_4| \leq 1$.

Lemma 2.2. (See Goodman, 1983) For a function $p \in \mathcal{P}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in D$$

the sharp inequality $|c_n| \leq 2$ holds for each $n \geq 1$ and

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Equality holds for the function $p(z) = \frac{1+z}{1-z}$.

Lemma 2.3. (See Efraimidis, 2016) Let $p \in \mathcal{P}$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$, and $\mu \in \mathbb{C}$. Then

$$|c_n - \mu c_k c_{n-k}| \leq 2 \max\{1, |2\mu - 1|\}, \quad 1 \leq k \leq n - 1.$$

If $|2\mu - 1| \geq 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$

or its rotations.

If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$

or its rotations.

Lemma 2.4. (See Arif, Tang, Hussain & Khan, 2019) Let $p \in \mathcal{P}$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in D$, and $\alpha, \beta, \gamma \in \mathbb{R}$. Then

$$\begin{aligned} |\alpha c_1^3 - \beta c_1 + \gamma c_3| &\leq 2|\alpha| + 2|\beta - 2\alpha| \\ &+ 2|\alpha - \beta + \gamma|. \end{aligned}$$

Lemma 2.5. (See Choi, Kim & Sugawa, 2007) Let $\bar{D} := \{\rho \in \mathbb{C} : |\rho| \leq 1\}$, and for $J, K, L \in \mathbb{R}$, let

$$Y(J, K, L) := \max\{|J + K\rho + L\rho^2| + 1 - |\rho|^2 : \rho \in \bar{D}\}. \quad (2.5)$$

If $JL \geq 0$, then

$$\begin{aligned} Y(J, K, L) &= \begin{cases} |J| + |K| + |L|, & |K| \geq 2(1 - |L|), \\ 1 + |J| + \frac{K^2}{4(1 - |L|)}, & |K| < 2(1 - |L|). \end{cases} \end{aligned}$$

If $JL < 0$, then

$$Y(J, K, L) = \begin{cases} 1 - |J| + \frac{K^2}{4(1 - |L|)}, & -4JL(L^2 - 1) \leq K^2 \wedge |K| < 2(1 - |L|), \\ 1 + |J| + \frac{K^2}{4(1 - |L|)}, & K^2 < \min\{4(1 + |L|)^2, -4JL(L^2 - 1)\} \\ R(J, K, L), & \text{otherwise,} \end{cases}$$

where

$$R(J, K, L) = \begin{cases} |J| + |K| - |L|, & |L|(|K| + 4|J|) \leq |JK|, \\ 1 + |J| + \frac{K^2}{4(1 - |L|)}, & |JK| < |L|(|K| - 4|J|), \\ |L| + |J| \sqrt{1 - \frac{K^2}{4JL}}, & \text{otherwise.} \end{cases}$$

3. Hankel Determinant with Logarithmic Coefficients for the Class $\mathcal{CST}_0(g)$

Theorem 3.1. If $f \in \mathcal{CST}_0(g)$ is of the form (1.1), then

$$|\gamma_1| \leq 1, |\gamma_2| \leq 1, |\gamma_3| \leq 1, |\gamma_4| \leq 3, |\gamma_5| \leq 7.$$

All inequalities are sharp.

Proof. By (1.4) there exists $p \in \mathcal{P}$ of the form (1.5) such that

$$\begin{aligned} \frac{f(z)}{z} &= p(z) \Rightarrow f(z) \\ &= zp(z). \end{aligned} \quad (3.1)$$

Substituting the series (1.1) and (1.5) into (3.1), then equating the coefficients, we get

$$\begin{aligned} a_2 &= c_1, a_3 = c_2, a_4 = c_3, a_5 \\ &= c_4. \end{aligned} \quad (3.2)$$

The inequality $|\gamma_1| \leq 1$ follows directly from (2.1), (3.2) and (1.12) with sharpness for the function f given by (3.1), where p is as in (1.5), also we can prove that inequality $|\gamma_1| \leq 1$ by using (1.12), (3.2) and Lemma (2.2).

Substituting (2.1) and (2.2) into (3.2) and from (1.13) it follows that

$$\begin{aligned} |\gamma_2| &= \frac{1}{2} \left| a_3 - \frac{1}{2} a_2^2 \right| \\ &= \frac{1}{2} \left| c_2 - \frac{1}{2} c_1^2 \right| \\ &= \frac{1}{2} |2\varepsilon_1^2 + 2(1 - \varepsilon_1^2)\varepsilon_2 - 2\varepsilon_1^2| \\ &= |(1 - \varepsilon_1^2)\varepsilon_2| \leq 1, \end{aligned}$$

with sharpness for the function f given by (2.3), where p is as in (1.5) with

$$\varepsilon_1 = 0 \text{ and any } \varepsilon_2 \in T.$$

Also, we can prove that $|\gamma_2| \leq 1$ using (1.13), (3.2) and Lemma (2.3).

Substituting (2.1), (2.2) and (2.3) into (3.2) and from (1.14) it follows that

$$\begin{aligned} |\gamma_3| &= \frac{1}{2} \left| a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right| = \frac{1}{2} \left| c_3 - c_1 c_2 + \frac{1}{3} c_1^3 \right| \\ &= \frac{1}{6} |c_1^3 - 3c_1 c_2 + 3c_3| \\ &= \frac{1}{6} |6\varepsilon_1^3 + 12(1 - \varepsilon_1^2)\varepsilon_1 \varepsilon_2 \\ &\quad - 6(1 - \varepsilon_1^2)\varepsilon_1 \varepsilon_2^2 \\ &\quad + 6(1 - \varepsilon_1^2)(1 - |\varepsilon_2|^2)\varepsilon_3 \\ &\quad - 6\varepsilon_1(2\varepsilon_1^2 + 2(1 - \varepsilon_1^2)\varepsilon_2) + 8\varepsilon_1^3| \\ &= \frac{1}{6} |2\varepsilon_1^3 - 6(1 - \varepsilon_1^2)\varepsilon_1 \varepsilon_2^2 \\ &\quad + 6(1 - \varepsilon_1^2)(1 - |\varepsilon_2|^2)\varepsilon_3| \\ &\leq \frac{1}{6} |2\varepsilon_1^3 - 6(1 - \varepsilon_1^2)\varepsilon_1 \varepsilon_2^2 \\ &\quad + 6(1 - \varepsilon_1^2)(1 - |\varepsilon_2|^2)|, \end{aligned}$$

where $\varepsilon_i \in D$, $i = 1, 2, 3$. Thus by setting $x := |\varepsilon_1| \in [0, 1]$ and $y := |\varepsilon_2| \in [0, 1]$,

we obtain

$$\begin{aligned} |\gamma_3| &\leq \frac{1}{6} [2x^3 + 6(1 - x^2)(1 - (1 + x)y^2)] \\ &= \frac{1}{6} [2x^3 - 6x^2 - 6(1 - x^2)(1 + x)y^2 + 6] \leq 1, (x, y) \in [0, 1] \times [0, 1]. \end{aligned}$$

Also we can prove that $|\gamma_3| \leq 1$ by using (1.14), (3.2) and Lemma (2.4).

$$\begin{aligned} |\gamma_3| &= \frac{1}{6} |c_1^3 - 3c_1 c_2 + 3c_3| \\ &\leq \frac{1}{6} (2|1| + 2|3 - 2| + |1 - 3 + 3|) \\ &\leq 1, \end{aligned}$$

where $\alpha = 1, \beta = 3, \gamma = 3$.

Substituting (3.2) into (1.15), then using triangle inequality and Lemmas (2.3) and (2.4), we get

$$\begin{aligned} |\gamma_4| &= \frac{1}{2} \left| c_4 - c_1 c_3 + c_1^2 c_2 - \frac{1}{2} c_2^2 - \frac{1}{4} c_1^4 \right| \\ &\leq \frac{1}{2} \left[\left| c_4 - \frac{1}{2} c_2^2 \right| + |c_1| \left| c_3 - c_1 c_2 + \frac{1}{4} c_1^3 \right| \right] \\ &\leq \frac{1}{2} \left(2 \left[2 \left| \frac{1}{4} \right| + 2 \left| 1 - \frac{1}{2} \right| + 2 \left| \frac{1}{4} - 1 + 1 \right| \right] \right) \\ &= \frac{1}{2} \left(2 + 2 \left(\frac{1}{2} + 1 + \frac{1}{2} \right) \right) = 3, \end{aligned}$$

where $\alpha = \frac{1}{4}, \beta = 1, \gamma = 1, \mu = \frac{1}{2}$.

Substituting (3.2) into (1.16), then using triangle inequality and Lemmas (2.2), (2.3) and (2.4), we get

$$|\gamma_5| = \frac{1}{2} \left| c_5 - c_1 c_4 - c_2 c_3 + c_1^2 c_3 + c_1 c_2^2 - c_2 c_1^3 + \frac{1}{5} c_1^5 \right|$$

$$\begin{aligned}
&= \frac{1}{2} \left| (c_5 - c_1 c_4) + c_1^2 \left(\frac{1}{5} c_1^3 - c_1 c_2 + c_3 \right) - c_2 (c_3 - c_1 c_2) \right| \\
&\leq \frac{1}{2} \left[|c_5 - c_1 c_4| + |c_2| |c_3 - c_1 c_2| \right. \\
&\quad \left. + |c_1|^2 \left| \frac{1}{5} c_1^3 - c_1 c_2 + c_3 \right| \right] \\
&\leq \frac{1}{2} \left[2 \right. \\
&\quad \left. + 4 \left(2 \left| \frac{1}{5} \right| + 2 \left| 1 - \frac{2}{5} \right| + 2 \left| \frac{1}{5} - 1 + 1 \right| \right) \right. \\
&\quad \left. + 4 \right] = \frac{1}{2} (6 + 4 \times 2) = 7,
\end{aligned}$$

where $\alpha = \frac{1}{5}$, $\beta = 1$, $\gamma = 1$, $\mu_1 = 1$, $\mu_2 = 1$.

Theorem 3.2: If $f \in \mathcal{CS}\mathcal{T}_0(g)$, is of the form (1.1), then

$$|H_{2,1}(F_f/2)| = |\gamma_1 \gamma_3 - \gamma_2^2| \leq 1.$$

The inequality is sharp.

Proof:

Substituting (1.12), (1.13), (1.14) and (3.2), we obtain

$$\gamma_1 \gamma_3 - \gamma_2^2 = \frac{1}{4} (a_2 a_4 - a_3^2 + \frac{1}{12} a_2^4) = \frac{1}{4} (c_1 c_3 - c_2^2 + \frac{1}{12} c_1^4).$$

By using lemma 2.1, we get

$$\begin{aligned}
\gamma_1 \gamma_3 - \gamma_2^2 &= \frac{1}{4} \left[2\varepsilon_1 (2\varepsilon_1^3 + 4(1 - \varepsilon_1^2)\varepsilon_1 \varepsilon_2 \right. \\
&\quad \left. - 2(1 - \varepsilon_1^2)\varepsilon_1 \varepsilon_2^2 \right. \\
&\quad \left. + 2(1 - \varepsilon_1^2)(1 - (|\varepsilon_2|)^2)\varepsilon_3 \right) \\
&\quad \left. - (2\varepsilon_1^2 + 2(1 - \varepsilon_1^2)\varepsilon_2)^2 + \frac{16}{12} \varepsilon_1^4 \right] \\
&= \frac{1}{4} \left[\frac{4}{3} \varepsilon_1^4 - 4(1 - \varepsilon_1^2)\varepsilon_2^2 \right. \\
&\quad \left. + 4(1 - \varepsilon_1^2)(1 - (|\varepsilon_2|)^2)\varepsilon_1 \varepsilon_3 \right]. \tag{3.3}
\end{aligned}$$

A. Suppose that $\varepsilon_1 = 1$. Then, by (3.3),

$$\begin{aligned}
\gamma_1 \gamma_3 - \gamma_2^2 &= \frac{1}{3} \Rightarrow |\gamma_1 \gamma_3 - \gamma_2^2| \\
&= \frac{1}{3}. \tag{3.4}
\end{aligned}$$

B. Suppose that $\varepsilon_1 = 0$. Then, by (3.3),

$$\begin{aligned}
\gamma_1 \gamma_3 - \gamma_2^2 &= \frac{1}{4} (-4\varepsilon_2^2) \Rightarrow |\gamma_1 \gamma_3 - \gamma_2^2| \\
&\leq 1. \tag{3.5}
\end{aligned}$$

C. Suppose that $\varepsilon_1 \in (0, 1)$. By the fact that $|\varepsilon_3| \leq 1$, from (3.3), we obtain

$$\begin{aligned}
|\gamma_1 \gamma_3 - \gamma_2^2| &\leq \frac{1}{4} \left[\left| 4(1 - \varepsilon_1^2)\varepsilon_1 \left(\frac{\varepsilon_1^3}{3(1 - \varepsilon_1^2)} - \frac{\varepsilon_2^2}{\varepsilon_1} \right. \right. \right. \\
&\quad \left. \left. + 1 - |\varepsilon_2|^2 \right) \right| \right] \\
&= (1 - \varepsilon_1^2)\varepsilon_1 \left| \frac{\varepsilon_1^3}{3(1 - \varepsilon_1^2)} \right. \\
&\quad \left. - \frac{1}{\varepsilon_1} \varepsilon_2^2 + 1 \right. \\
&\quad \left. - |\varepsilon_2|^2 \right|, \tag{3.6}
\end{aligned}$$

where $J = \frac{\varepsilon_1^3}{3(1 - \varepsilon_1^2)}$, $K = 0$, $L = -\frac{1}{\varepsilon_1}$.

Since $JL < 0$, we apply Lemma 2.5 only for the case II.

C1. Note that the inequality

$$\begin{aligned}
-4JL \left(\frac{1}{L^2} - 1 \right) - K^2 &= \frac{4\varepsilon_1^2}{3(1 - \varepsilon_1^2)} (\varepsilon_1^2 - 1) \\
&= -\frac{4\varepsilon_1^2}{3} < 0,
\end{aligned}$$

which evidently holds for $\varepsilon_1 \in (0, 1)$. Moreover, the inequality $|K| < 2(1 - |L|)$ is equivalent to $0 < 2(1 - \varepsilon_1)$, which is false for $\varepsilon_1 \in (0, 1)$.

C2. Since

$$\begin{aligned}
4(1 + |L|)^2 &= 4 \left(1 + \frac{1}{\varepsilon_1} \right)^2 = 4 \left(\frac{\varepsilon_1 + 1}{\varepsilon_1} \right)^2 > 0, \\
-4JL \left(\frac{1}{L^2} - 1 \right) &= -\frac{4\varepsilon_1^2}{3} < 0,
\end{aligned}$$

we see that the inequality

$$\begin{aligned}
K^2 = 0 &< \min \left\{ 4(1 + |L|)^2, -4JL \left(\frac{1}{L^2} - 1 \right) \right\} \\
&= -\frac{4\varepsilon_1^2}{3}
\end{aligned}$$

is false for $\varepsilon_1 \in (0, 1)$.

C3. Observe that the inequality

$$|L|(|K| + 4|J|) - |JK| = \frac{4\varepsilon_1^2}{3(1 - \varepsilon_1^2)} \leq 0,$$

is equivalent to $\varepsilon_1^2 \leq 0$, which is false for $\varepsilon_1 \in (0, 1)$.

C4. Note that the inequality

$$|JK| - |L|(|K| - 4|J|) = \frac{4\varepsilon_1^2}{3(1 - \varepsilon_1^2)} \leq 0,$$

is equivalent to $\varepsilon_1^2 \leq 0$, which is false for $\varepsilon_1 \in (0, 1)$.

C5. It remains to consider the last case in Lemma 2.5 and taking into account C4 holds for $\varepsilon' < \varepsilon_1 < 1$.

$$|L| + |J| \sqrt{1 - \frac{K^2}{4JL}} = \frac{1}{\varepsilon_1} + \frac{\varepsilon_1^3}{3(1 - \varepsilon_1^2)} = \frac{\varepsilon_1^3 - 3\varepsilon_1^2 + 3}{3\varepsilon_1(1 - \varepsilon_1^2)}.$$

Then, by (3.3),

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq (1 - \varepsilon_1^2)\varepsilon_1 \left(\frac{\varepsilon_1^3 - 3\varepsilon_1^2 + 3}{3\varepsilon_1(1 - \varepsilon_1^2)} \right) = \frac{\varepsilon_1^3 - 3\varepsilon_1^2 + 3}{3},$$

when $\varepsilon_1 = 0$,

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq 1.$$

Theorem 3.3: If $f \in \mathcal{CS}\mathcal{T}_0(g)$, is of the form (1.1), then

$$\left| H_{2,2} \left(\frac{F_f}{2} \right) \right| = |\gamma_2\gamma_4 - \gamma_3^2| \leq 19.$$

The inequality is sharp.

Proof: Substituting (1.13), (1.14) and (1.14) into (3.2) we obtain

$$\begin{aligned} |\gamma_2\gamma_4 - \gamma_3^2| &= \left| -\frac{3}{4}a_2a_3a_4 + \frac{1}{4}a_3a_5 + \frac{1}{16}a_2^2a_3^2 - \frac{1}{8}a_3^3 \right. \\ &\quad \left. - \frac{17}{48}a_3a_2^4 - \frac{1}{8}a_5a_2^2 - \frac{1}{24}a_4a_2^3 - \frac{1}{4}a_4^2 \right. \\ &\quad \left. + \frac{1}{288}a_2^6 \right| \\ &= \left| -\frac{3}{4}c_1c_2c_3 + \frac{1}{4}c_2c_4 + \frac{1}{16}c_1^2c_2^2 - \frac{1}{8}c_2^3 \right. \\ &\quad \left. - \frac{17}{48}c_2c_1^4 - \frac{1}{8}c_4c_1^2 - \frac{1}{24}c_3c_1^3 - \frac{1}{4}c_3^2 \right. \\ &\quad \left. + \frac{1}{288}c_1^6 \right|. \end{aligned}$$

By using triangle inequality and Lemma (2.2), (2.3) and (2.4), we get

$$\begin{aligned} |\gamma_2\gamma_4 - \gamma_3^2| &= \left| c_3 \left(-\frac{1}{24}c_1^3 - \frac{3}{4}c_1c_2 - \frac{1}{4}c_3 \right) \right. \\ &\quad \left. + \frac{1}{4}c_4 \left(c_2 - \frac{1}{2}c_1^2 \right) \right. \\ &\quad \left. - \frac{17}{48}c_1^4 \left(c_2 - \frac{1}{102}c_1^2 \right) \right. \\ &\quad \left. - \frac{1}{8}c_2^2 \left(c_2 - \frac{1}{2}c_1^2 \right) \right| \\ &\leq |c_3| \left| -\frac{1}{24}c_1^3 - \frac{3}{4}c_1c_2 - \frac{1}{4}c_3 \right| \\ &\quad + \frac{1}{4}|c_4| \left| c_2 - \frac{1}{2}c_1^2 \right| \\ &\quad + \frac{17}{48}|c_1|^4 \left| c_2 - \frac{1}{102}c_1^2 \right| \\ &\quad + \frac{1}{8}|c_2|^2 \left| c_2 - \frac{1}{2}c_1^2 \right| \\ &\leq 2 \left(2 \left| -\frac{1}{24} \right| + 2 \left| \frac{3}{4} + \frac{1}{12} \right| \right. \\ &\quad \left. + 2 \left| -\frac{1}{24} - \frac{3}{4} + \frac{1}{4} \right| \right) + 1 + \frac{17}{48} \times 16 \times 2 \\ &\quad + 1 = 2 \left(\frac{14}{12} + \frac{5}{3} \right) + 2 + \frac{34}{3} \\ &= \frac{17}{3} + \frac{34}{3} + 2 = 19, \end{aligned}$$

where

$$\alpha = \frac{-1}{24}, \beta = \frac{-3}{4}, \gamma = \frac{-1}{4}, \mu_1 = \frac{1}{2}, \mu_2 = \frac{1}{102}, \mu_3 = \frac{1}{2}.$$

Theorem 3.4: If $f \in \mathcal{CS}\mathcal{T}_0(g)$, is of the form (1.1), then

$$|\gamma_1\gamma_4 - \gamma_2\gamma_3| \leq \frac{10}{3}.$$

The inequality is sharp.

Proof: Substituting (1.12), (1.13), (1.14) and (1.15) into (3.2) we get

$$\begin{aligned} |\gamma_1\gamma_4 - \gamma_2\gamma_3| &= \left| \frac{1}{4}a_2a_5 - \frac{1}{8}a_2^2a_4 + \frac{1}{24}a_3a_2^3 + \frac{1}{8}a_3^2a_2 \right. \\ &\quad \left. + \frac{1}{48}a_2^5 \right| \\ &= \left| \frac{1}{4}c_1c_4 - \frac{1}{8}c_1^2c_3 + \frac{1}{24}c_2c_3^3 + \frac{1}{8}c_2^2c_1 \right. \\ &\quad \left. + \frac{1}{48}c_1^5 \right| \\ &= \left| c_1^2 \left(\frac{1}{48}c_1^3 + \frac{1}{24}c_1c_2 - \frac{1}{8}c_3 \right) \right. \\ &\quad \left. + \frac{1}{4}c_1 \left(c_4 + \frac{1}{2}c_2^2 \right) \right|. \end{aligned}$$

By using triangle inequality and Lemma (2.2), (2.3) and (2.4), we get

$$\begin{aligned}
|\gamma_1\gamma_4 - \gamma_2\gamma_3| &\leq 4 \left(2 \left| \frac{1}{48} \right| + 2 \left| -\frac{1}{24} - \frac{1}{24} \right| \right. \\
&\quad \left. + 2 \left| \frac{1}{48} + \frac{1}{24} - \frac{1}{8} \right| \right) \\
&\quad + \frac{1}{4} \times 2 \\
&\quad \times 2 \max\{1, 2\} = 4 \left(\frac{1}{24} + \frac{1}{6} + \frac{1}{8} \right) + 2 \\
&= \frac{10}{3},
\end{aligned}$$

where $\alpha = \frac{1}{48}$, $\beta = \frac{-1}{24}$, $\gamma = \frac{-1}{8}$, $\mu = \frac{-1}{2}$.

Theorem 3.5: If $f \in \mathcal{CST}_0(g)$, is of the form (1.1), then

$$|H_{3,1}(F_f/2)| \leq 36.$$

The inequality is sharp.

Proof: $H_{3,1}(F_f/2) = \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{vmatrix}$

$$= \gamma_3 H_{2,2}(F_f/2) - \gamma_4(\gamma_1\gamma_4 - \gamma_2\gamma_3) + \gamma_5 H_{2,1}(F_f/2)$$

$$|H_{3,1}(F_f/2)| = |\gamma_3 H_{2,2}(F_f/2) - \gamma_4(\gamma_1\gamma_4 - \gamma_2\gamma_3) + \gamma_5 H_{2,1}(F_f/2)|.$$

By using triangle inequality and theorems (3.1), (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
|H_{3,1}(F_f/2)| &\leq |\gamma_3| |H_{2,2}(F_f/2)| + |\gamma_4| |\gamma_1\gamma_4 - \gamma_2\gamma_3| \\
&\quad + |\gamma_5| |H_{2,1}(F_f/2)| \\
&= 1 \times 19 + 3 \times \frac{10}{3} + 7 \times 1 = 36.
\end{aligned}$$

4. Zalcman Functional

Lecko and Sim (2019) obtained the sharp upper bound of the Zalcman functional $J_{2,3}(f) = a_4 - a_2a_3$ being a special case of the generalized Zalcman functional $J_{n,q}(f) = a_{n+q-1} - a_n a_q$, $n, q \in \mathbb{N} \setminus \{1\}$ for functions from $\mathcal{ST}(1)$. We will find also the sharp bounds of the Zalcman functionals $J_{3,3}(f)$ for the family $\mathcal{CST}_0(g)$.

Theorem 4.1. If $f \in \mathcal{CST}_0(g)$ is of the form (1.1), then

$$J_{3,3}(f) = |a_5 - a_3^2| \leq 2.$$

Proof. From (3.2) by using (2.2) and (2.4) it follows that

$$\begin{aligned}
J_{3,3}(f) &= |a_5 - a_3^2| = |c_4 - c_2^2| \\
&= \left| -\frac{1}{8}c_1^4 \right. \\
&\quad + \frac{1}{8}\varepsilon_2 c_1^2 (4 - c_1^2)(\varepsilon_2^2 - 7\varepsilon_2 + 3) \\
&\quad + \frac{1}{4}\varepsilon_2^2 (4 - c_1^2)(c_1^2 - 2) \\
&\quad - \frac{1}{2}(4 - c_1^2)(1 - |\varepsilon_2|^2)[c_1(\varepsilon_2 - 1)\varepsilon_3 \\
&\quad \left. + \varepsilon_2\varepsilon_3^2 - (1 - |\varepsilon_3|^2)\varepsilon_4] \right|. \quad (4.1)
\end{aligned}$$

Suppose that

$$v_1(c_1, \varepsilon_2) = -\frac{1}{8}c_1^4 + \frac{1}{8}\varepsilon_2 c_1^2 (4 - c_1^2)(\varepsilon_2^2 - 7\varepsilon_2 + 3) + \frac{1}{4}\varepsilon_2^2 (4 - c_1^2)(c_1^2 - 2),$$

$$v_2(c_1, \varepsilon_2)\varepsilon_3 = -\frac{1}{2}c_1(4 - c_1^2)(1 - |\varepsilon_2|^2)(\varepsilon_2 - 1)\varepsilon_3,$$

$$v_3(c_1, \varepsilon_2)\varepsilon_3^2 = -\frac{1}{2}(4 - c_1^2)(1 - |\varepsilon_2|^2)\varepsilon_2\varepsilon_3^2,$$

$$v_4(c_1, \varepsilon_2)\varepsilon_4 = \frac{1}{2}(4 - c_1^2)(1 - |\varepsilon_2|^2)(1 - |\varepsilon_3|^2)\varepsilon_4.$$

Taking $|\varepsilon_2| = x$, $|\varepsilon_3| = y$. In light of $|\varepsilon_4| \leq 1$, we have

$$\begin{aligned}
|c_4 - c_2^2| &\leq |v_1(c_1, x)| + |v_2(c_1, x)|y + |v_3(c_1, x)|y^2 \\
&\quad + |v_4(c_1, x, y)| \\
&\leq \psi(c_1, x, y), \quad (4.2)
\end{aligned}$$

where $\psi(c_1, x, y) = u_1(c_1, x) + u_2(c_1, x)y + u_3(c_1, x)y^2 + u_4(c_1, x)(1 - y^2)$,

and

$$\begin{aligned}
u_1(c_1, x) &= \frac{1}{8}c_1^4 + \frac{1}{8}xc_1^2(4 - c_1^2)(x^2 - 7x + 3) \\
&\quad + \frac{1}{4}x^2(4 - c_1^2)(c_1^2 - 2),
\end{aligned}$$

$$u_2(c_1, x)y = \frac{1}{2}c_1(4 - c_1^2)(1 - x^2)(x - 1)y,$$

$$u_3(c_1, x)y^2 = \frac{1}{2}(4 - c_1^2)(1 - x^2)xy^2,$$

$$u_4(c_1, x)(1 - y^2) = \frac{1}{2}(4 - c_1^2)(1 - x^2)(1 - y^2).$$

Now, we have to find the maximum value in $\Omega: [0, 2] \times [0, 1] \times [0, 1]$. We determine the maximum value of the function $\psi(c_1, x, y)$ in the interior of Ω on the edges and vertices of Ω .

I. First, we consider the interior points of Ω . Differentiating (4.2) with respect to y and

after some simplifications, we obtain

$$\begin{aligned}
\frac{\partial \psi}{\partial y} &= \frac{1}{2}(4 - c_1^2)(1 - x^2)(x - 1)(c_1 + 2y) \\
&\neq 0. \quad (4.3)
\end{aligned}$$

Therefore, the function has no maximum value in $(0, 2) \times (0, 1) \times (0, 1)$.

II. Next, we discuss the optimum value in the interior of six faces of Ω .

On the face $c_1 = 0$, $\psi(c, x, y)$ reduces to

$$\begin{aligned}
\kappa_1(x, y) &= \psi(0, x, y) \\
&= 2xy^2 - 2x^3y^2 - 4x^2 - 2y^2 + 2x^2y^2 + 2. \quad (4.4)
\end{aligned}$$

The function κ_1 has no maxima in $(0,1) \times (0,1)$ since

$$\begin{aligned}\frac{\partial \kappa_1}{\partial x} &= 2y^2 - 6x^2y^2 - 8x \\ &+ 4xy^2 \\ \frac{\partial \kappa_1}{\partial y} &= 4xy - 4x^3y - 4y + 4x^2y.\end{aligned}$$

By implementing numerical methods, we see that the system $\frac{\partial \kappa_1}{\partial x} = 0$ and $\frac{\partial \kappa_1}{\partial y} = 0$

has no solution in $(0,1) \times (0,1)$.

On $c_1 = 2$, $\psi(c_1, x, y)$ takes the form

$$\begin{aligned}\psi(2, x, y) &= 2, x, y \\ &\in (0,1).\end{aligned}\quad (4.5)$$

On $x = 0$, $\psi(c_1, x, y)$ can be written as

$$\begin{aligned}\psi(c_1, 0, y) &= \kappa_2(c_1, y) \\ &= \frac{1}{8}c_1^4 + \frac{1}{2}(4 - c_1^2)(1 - y^2 - c_1y), c_1 \\ &\in (0,1).\end{aligned}\quad (4.6)$$

This implies that the function κ_2 has maxima $(0,2) \times (0,1)$, since

$$\begin{aligned}\frac{\partial \kappa_2}{\partial y} &= -\frac{1}{2}(4 - c_1^2)(c + y), \\ \frac{\partial \kappa_2}{\partial c_1} &= \frac{1}{2}c_1^3 - \frac{1}{2}(4 - c_1^2)y - c_1(1 - y^2 - cy).\end{aligned}$$

Using numerical methods, we see that the system $\frac{\partial \kappa_2}{\partial y} =$

$$0 \text{ and } \frac{\partial \kappa_2}{\partial c_1} = 0$$

has no solution in $(0,2) \times (0,1)$.

On $x = 1$, $\psi(c_1, x, y)$ reduces to

$$\begin{aligned}\kappa_3(c, y) &:= \psi(c_1, 1, y) \\ &= \frac{1}{8}c^4 - \frac{3}{8}c_1^2(4 - c_1^2) \\ &+ \frac{1}{4}(4 - c_1^2)(c_1^2 - 2).\end{aligned}\quad (4.7)$$

There is no critical point for the function κ_3 in $(0,2) \times (0,1)$, since

$$\begin{aligned}\frac{\partial \kappa_3}{\partial c_1} &= 2c_1^3 - 3c_1 - c_1^3 + 3c_1 = c_1^3 \neq 0, c_1 \\ &\in (0,2).\end{aligned}$$

On $y = 0$, $\psi(c, x, y)$ reduces to

$$\begin{aligned}\kappa_4(c_1, x) &= \psi(c_1, x, 0) \\ &= \frac{1}{8}c_1^4 \\ &+ \frac{1}{8}xc_1^2(4 - c_1^2)(x^2 - 7x + 3) \\ &+ \frac{1}{4}x^2(4 - c_1^2)(c_1^2 - 2) \\ &+ \frac{1}{2}(4 - c_1^2)(1 - x^2),\end{aligned}\quad (4.8)$$

and

$$\frac{\partial \kappa_4}{\partial x} = \frac{1}{2}(4 - c_1^2)\left(\frac{3}{4}c_1^2 - \frac{5}{2}c_1^2x + \frac{3}{4}c_1^2x^2 - 4x\right).$$

Using numerical methods, we see that the system $\frac{\partial \kappa_4}{\partial c_1} =$

$$0 \text{ and } \frac{\partial \kappa_4}{\partial x} = 0$$

has no solution in $(0,2) \times (0,1)$.

On $y = 1$, the function $\psi(c_1, x, y)$ takes the form

$$\begin{aligned}\kappa_5(c_1, x) &= \frac{1}{8}c_1^4 + \frac{1}{8}xc_1^2(4 - c_1^2)(x^2 - 7x + 3) \\ &+ \frac{1}{4}x^2(4 - c_1^2)(c_1^2 - 2) \\ &= +\frac{1}{2}c_1(4 - c_1^2)(1 - x^2)(x - 1) \\ &= \frac{1}{8}c_1^4 + \frac{1}{2}(4 - c_1^2)\left(\frac{1}{4}c_1^2x^3 - \frac{5}{4}c_1^2x^2 + \frac{3}{4}c_1^2x - x^2\right. \\ &\quad \left.+ c_1x - c_1x^3 + c_1x^2 - c_1\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial \kappa_5}{\partial c_1} &= \frac{1}{2}c_1^3 - c_1\left(\frac{1}{4}c_1^2x^3 - \frac{5}{4}c_1^2x^2 + \frac{3}{4}c_1^2x - x^2\right. \\ &\quad \left.+ c_1x - c_1x^3 + c_1x^2 - c_1\right) \\ &+ \frac{1}{2}(4 - c_1^2)\left(\frac{1}{2}c_1x^3 - \frac{5}{2}c_1x^2\right. \\ &\quad \left.+ \frac{3}{2}c_1x + x - x^3 + x^2 - 1\right), \\ \frac{\partial \kappa_5}{\partial x} &= \frac{1}{2}(4 - c_1^2)\left(\frac{3}{4}c_1^2x^2 - \frac{5}{2}c_1^2x + \frac{3}{4}c_1^2 - 2x + c_1\right. \\ &\quad \left.- 3c_1x^2 + 2c_1x\right).\end{aligned}$$

Similarly, on the face $y = 0$, the system of equations $\frac{\partial \kappa_5}{\partial c_1} =$

$$0 \text{ and } \frac{\partial \kappa_5}{\partial x} = 0$$

has no solution in $(0,2) \times (0,1)$.

III. Now, we investigate the maximum of $\psi(c_1, x, y)$ on the edges of Ω . From (4.6), we obtain $\psi(c_1, 0, 0) =: s_1(c_1) = \frac{1}{8}c_1^4 + \frac{1}{2}(4 - c_1^2)$.

Since

$$s_1(c_1) = \frac{1}{2}c_1^3 - c_1 = 0.$$

then there is one critical point $c_1 = \sqrt{2}$ for $c_1 \in [0, 2]$. Therefore, s_1 is increasing in $[\sqrt{2}, 2]$ and hence a maximum is achieved at $c_1 = 2$.

Therefore

$$\psi(c_1, 0, 0) \leq 2.$$

Also, from (4.6) at $y = 1$, we write

$$\psi(c_1, 0, 1) = s_2(c_1) = \frac{1}{8}c_1^4 - \frac{1}{2}(4 - c_1^2)(c_1 - 1).$$

$$\text{Since } \dot{s}_2(c_1) = \frac{1}{2}c_1^3 + \frac{3}{2}c_1^2 - c_1 - 1 = 0.$$

By implementing numerical methods, we get $c_1 = 1$ in $[0, 2]$.

Thus

$$\psi(c_1, 0, 1) \leq \frac{1}{8} = 0.125.$$

Putting $c_1 = 0$ in (4.6), we have

$$\psi(0, 0, y) = 2(1 - y^2).$$

Since (4.7) is independent of x , we obtain

$$\begin{aligned} \psi(c_1, 1, 1) &= \psi(c_1, 1, 0) = s_3(c_1) \\ &:= \frac{1}{8}c_1^4 - \frac{3}{8}c_1^2(4 - c_1^2) \\ &\quad + \frac{1}{4}(4 - c_1^2)(c_1^2 - 2) \end{aligned}$$

Since $\dot{s}_3(c_1) = c_1^3 > 0$ for $[0, 2]$, $s_3(c_1)$ is increasing in $[0, 2]$ and hence a maximum is achieved at $c_1 = 2$.

Thus

$$\psi(c_1, 1, 1) = \psi(c_1, 1, 0) \leq 2.$$

Putting $c_1 = 0$ in (4.7), we obtain

$$\psi(0, 1, y) \leq 2.$$

Equation (4.5) is independent of variables x and y . Thus,

$$\psi(2, x, 0) = \psi(2, x, 1) = \psi(2, 0, y) = \psi(2, 1, y) = 2, \quad x, y \in [0, 1].$$

from equation (4.4), we have

$$\psi(0, x, 0) = s_4(x) = -4x^2 + 2.$$

Since $\dot{s}_4(x) = -8x < 0$ for $[0, 1]$, $s_4(x)$ is decreasing in $[0, 1]$ and hence a maximum is achieved at $x = 0$. Thus,

$$\psi(0, x, 0) = 2,$$

$$\psi(0, x, 1) = s_5(x) = -2x^3 - 2x^2 + 2x.$$

Since $\dot{s}_5(x) = -6x^2 - 4x + 2 < 0$ for $[0, 1]$, $s_5(x)$ is decreasing in $[0, 1]$ and hence a maximum is achieved at $x = \frac{1}{3}$. Thus,

$$\psi(0, x, 1) = \frac{10}{27} \approx 0.37037.$$

This completes the result.

5. Conclusion

Due to the great importance of logarithmic coefficients, Kowalczyk et al. (2018) proposed the topic of studying the Hankel determinant with the entry of logarithmic coefficients. In our present investigation, we successfully investigated a subclass of close-to-star functions, focusing on the class $\mathcal{CST}_0(g)$ where $g(z) = z$. we have obtained estimate for some initial logarithmic coefficients up to γ_5 , as well as $H_{2,1}, H_{2,2}$ and $H_{3,1}$ with entries involving logarithmic coefficients and some related inequalities problems on logarithmic coefficients. The third Hankel determinant bound for the logarithmic coefficients as the entry for this class were determined.

The bounds of various coefficient functionals in the class presented in this paper were obtained due by connecting this class with the Carathéodory class. It is worth noting that a thorough understanding of the Carathéodory class, including estimates of coefficient functionals, is a good tool in studies of other classes of analytic functions. Using these results, one can easily obtain the fourth and fifth Hankel determinants.

Furthermore, we examined the Zalcman functional $J_{3,3}(f)$ and established a sharp bound of 2. This was supported by a detailed three-dimensional parameter analysis, ensuring the accuracy and sharpness of the result.

In this paper, we have benefited from several recent studies. However, there are some key differences between our work and theirs. Below are some of the main distinctions:

Comparison with Guo et al. (2024):

1. Class of functions studies

- This paper focuses on the subclass $\mathcal{CST}_0(g)$, where $g(z) = z$. This imposes the condition $\operatorname{Re} \frac{f(z)}{z} > 0$, which connects the class to the Carathéodory class $\mathcal{P}$.
- Guo et al. consider a broader class denoted as $\mathcal{ST}(1)$ (a subclass of close-to-star functions), without restricting to a specific, which makes their setting slightly more general but less specific in analytic structure.

2. All the main results in this paper are distinct from those in paper by Guo et al. because the two classes studied are different.

Comparison with Lecko and Sim (2019):

1. Class of functions studies

- This paper focuses on the subclass $\mathcal{CST}_0(g)$, where $g(z) = z$. This imposes the condition $\operatorname{Re} \frac{f(z)}{z} > 0$, which connects the class to the Carathéodory class $\mathcal{P}$.

- Lecko and Sim focus on the two subclasses $\mathcal{ST}(1)$ and $\mathcal{ST}(i)$ (subclasses of close – to – star functions), which connect the class to the Carathéodory class $\mathcal{P}$.
- 2. All the main results in this paper are distinct from those in paper by Lecko and Sim because the two classes studied are different.
- 3. This paper focuses on estimating logarithmic coefficients up to γ_5 , computing Hankel determinants $H_{2,1}$, $H_{2,2}$ and $H_{3,1}$, and analyzing Zalcman functionals such as $\mathcal{J}_{3,3}(f)$, while Lecko and Sim focus on estimating logarithmic coefficients up to γ_3 , computing Hankel determinants $H_{2,2}$, and analyzing Zalcman functionals such as $\mathcal{J}_{2,3}(f)$.

There are significant future challenges concerning the logarithmic coefficients from γ_6 onwards due to the increasing number of terms. We have derived the expression for γ_6 , but we were not able to establish a specific theorem for it, especially for the subclasses of close-to-star functions. To the best of our knowledge, no one has achieved that. However, God willing, in our future work, we plan to incorporate the γ_6 coefficient we have obtained into several theorems.

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المحددات من نوع هانكل H_2 و H_3 للمعاملات اللوغاريتمية لصنف جزئي من الدوال القريبة من النجمية

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ملخص

تبحث هذه الورقة المعاملات اللوغاريتمية لصنف جزئي من الدوال القريبة من النجمية، نحن نثبت الحدود الدقيقة لمعاملات هانكل، محدّدات هانكل $H_{2,2}, H_{2,1}, H_{3,1}$ و $H_{3,1}$ المرتبطة بالصنف $(g)_0 \llbracket CST \rrbracket$ حيث $g(z)=z$. بالإضافة الى ذلك، نحن نستنتج التقديرات الدقيقة لتابع ز الكمان $J_{3,3}(f)$ ضمن نفس الصنف. **الكلمات المفتاحية:** الدوال الأحادية، الدوال القريبة من النجمية، محددة هانكل، المعاملات اللوغاريتمية، تابع ز الكمان، صنف كاراثيودوري.

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